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D-brane charges in five-brane backgrounds

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Abstract: We discuss the discrete $\mathbb{Z}_k$ D-brane charges (twisted K-theory charges) in five-brane backgrounds from several different points of view. In particular, we interpret it as a result of a standard Higgs mechanism. We show that certain degrees of freedom (singletons) on the boundary of space can extend the corresponding $\mathbb{Z}_k$ symmetry to U(1). Related ideas clarify the role of AdS singletons in the AdS/CFT correspondence.

Keywords: M-Theory, D-branes, Discrete and Finite Symmetries.

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1. Introduction

D-brane charges and RR fluxes are classified by K-theory, at weak coupling [1, 2, 3]. It is important to bear in mind that all the arguments in favor of this rely on a picture of type II strings valid at weak coupling and on smooth spaces. Moreover, there is some tension between the K-theoretic classification of charges and fluxes and both U-duality (which mixes RR and NSNS degrees of freedom), and with the AdS/CFT correspondence (which treats charges and fluxes more democratically).

An important open problem is finding a more broadly applicable homotopy classification of both charges with fluxes. Even stating this open problem in a precise and useful way would constitute some progress.

With this motivation in mind the present paper studies the physical meaning of the K-theoretic charge group $K^*_H(X)$ in the presence of nontorsion H-flux, more specifically in backgrounds associated with 5-branes.[1] We will find that in some situations the K-theoretic classification can be slightly misleading, and we will show (in our examples) how to correct it.

One of the hallmarks of K-theoretic charges is that they naturally include torsion charges, and this is often cited as an important distinction from more traditional viewpoints on charges. Nevertheless, one should not lose sight of the fact that torsion charges also can arise naturally in standard gauge theories. It is true that the charges of a U(1) gauge group label the different representations and take values in Z. But, if the U(1) gauge symmetry is broken to the integers modulo k, denoted here by Z_k, through a standard Higgs mechanism via a charged scalar field of charge $k > 1$, then the unbroken gauge symmetry is Z_k. The dual group of representations is again Z_k leading to a torsion group of charges. We will see that this is precisely what happens in some examples of K-theoretic torsion charges. We will also see in these examples that there are often physical modes, zero-modes of RR potentials, which effectively restore the U(1) symmetry. In the context of string theory on AdS spaces these modes reside at the boundary and are sometimes called "singletons" (or "doubletons"). We will extend the terminology to a wider context and refer to these boundary modes as "singletons."

In appendix A we discuss general properties of p-forms with Chern Simons couplings and in appendix B we explain how the singleton is related to the U(1) subgroup of the gauge group in the AdS/CFT correspondence.

1.1 The general class of backgrounds

We will now be more precise about the class of backgrounds under consideration. We will focus on type IIA string theory with the action

$$
\begin{aligned}
S = {} & \frac{1}{(2\pi)^7\alpha'^4}\int\sqrt{-\det g}\,e^{-2\phi}(\mathcal{R}+4(\nabla\phi)^2) - \frac{1}{2(2\pi)^3\alpha'^2}\int e^{-2\phi}H\wedge *H - \\
& -\frac{1}{2(2\pi)^7\alpha'^3}\int R_2\wedge *R_2\frac{1}{2(2\pi)^3-\alpha'}\int R_4\wedge *R_4 + \frac{1}{4\pi}\int C_3\wedge d - C_3\wedge H\,.
\end{aligned}
$$

[1] It is an interesting question whether any background with cohomologically nontrivial H-flux can be regarded as one created by 5-branes.

Here g is the string metric, ϕ is the dilaton, $R_2 := dC_1$ is the RR 2-form fieldstrength, and $R_4 := dC_3 - H \wedge C_1$.[2] [3]

We take spacetime to be a product of time and 9-dimensional space $\mathbb{R}_t \times X_9$, where X_9 can be compact or noncompact. Let us assume we have a IIA geometry with a *static* metric on $\mathbb{R}_t \times X_9$ such that near infinity we have $X_9 = \mathbb{R} \times X_8$ ($\mathbb{R}$ is the radial direction), where X_8 is fibered by $S^3 \to X_8 \to X_5$. Moreover, we assume that S^3 is not a contractible cycle in X_9. The NS three form fieldstrength $H = k\xi_3$ for some 3-form ξ_3 with integral periods such that $\xi_3 \to \omega_3(1 + \mathcal{O}(1/r))$ at infinity, where ω_3 is the volume form of S^3, normalized to have period $= 1$. The Bianchi identity and the equation of motion imply that $d\xi_3 = 0$ and $d(*e^{-2\phi}\xi_3) = 0$. At infinity $H \to k\omega_3$.

The quintessential background with cohomologically nontrivial H-flux is the NS5-brane of Callan, Harvey and Strominger:

$$\begin{aligned} ds^2_{IIA} &= (dx)^2 + U(dy_\perp)^2 \\ e^{2\phi - 2\phi_\infty} &= U = 1 + \frac{k\alpha'}{r^2} \\ H &= -k\omega_3\,, \end{aligned} \tag{1.1}$$

where ω_3 is the integral class on S^3 and $r := |y_\perp|$. We will sometimes restrict attention to the throat region $e^{2\phi_\infty}\alpha' \ll r^2 \ll k\alpha'$. While this is the motivating example we often have in mind, the strong coupling singularity can lead to difficulties and ambiguities in our conclusions. We therefore wish to consider other examples where the string coupling is bounded above, although it can become zero as we approach the boundary of the spacetime.

Examples

1. The doubly Wick rotated near-extremal NS5-brane [6, 7][4]

$$\begin{aligned} ds^2_{string} &= (1 - r_0^2/r^2)d\phi^2 + \left(1 + \frac{Q_5}{r^2}\right)\left[\frac{dr^2}{1 - r_0^2/r^2} + r^2 ds^2_{S^3}\right] + (-dt^2 + ds_4^2) \\ e^{2\phi - 2\phi_\infty} &= 1 + \frac{Q_5}{r^2} \end{aligned}$$

[2] When the M-theory circle is a non-trivial fibration, or C_3 is not globally well-defined, or there is a Romans mass then the above expression must be modified.

[3] The topological terms in this action and in the M-theory action lead to a natural normalization convention for differential forms: The fieldstrength of the 4-form flux in M-theory should be dimensionless and should have periods which are 2π times an integral class (more precisely, an integral class plus $\frac{1}{4}p_1(TX)$ [4]). This fixes most of the conventions for the IIA lagrangian. In particular the H-flux has integral periods. In other words, our normalizations for RR fields differ from [5] in the following way: $C^{us}_{p+1} = \mu_p C^{\text{Polchinski}}_{p+1}$, where $\mu_p^{-2} = (4\pi^2\alpha')^p\alpha'$. Similarly, for the NS three form we have $(2\pi)^2\alpha' H_3^{us} = H_3^{\text{Polchinski}}$.

[4] This background has a tachyonic mode due to the negative specific heat of the euclidean black hole. It will be clear that this is not important for our discussion.

$$Q_5 = r_0^2 \sinh^2 \beta \,, \qquad \alpha' k = r_0^2 \frac{\sinh 2\beta}{2} \,. \tag{1.2}$$

2. The solution describing k $NS5$ branes wrapped on $S^2 \times R^{3,1}$ [8] . In this solution the structure of the spacetime near the boundary is the radial direction times $R^{1,3} \times X_5$, where X_5 is a (topologically trivial, but metrically nontrivial) S^3 fibration over S^2. The full geometry is topologically $R^{1,3} \times S^3 \times B^3$. In other words, the S^2 is filled by a three ball in the full geometry. The radius of S^3 is constant and there is a constant H flux equal to k over the three sphere. Even though in [8] the type IIB version was mainly considered, we can similarly consider the type IIA version of this background. This is a supersymmetric background preserving four supercharges.

3. Geometries of the form $AdS_3 \times S^3 \times M_4$, where M_4 can be $K3$, T^4 or $S^3 \times S^1$ with NS H fields on AdS_3 and S^3.

Unfortunately, it is not obvious how to incorporate the interesting solutions of Klebanov and Strassler [9] since they have RR fields and we do not know what replaces K-theory in the presence of RR fields.

In order to measure the RR charge present in these backgrounds we should measure the flux of the RR fields at infinity, i.e. at the boundary ∂X_9. As argued in [3, 10, 11, 12] RR fieldstrengths are topologically classified by K-theory, so the fields at the boundary are classified by

$$\begin{aligned} \mathcal{Q}^{RR}_{IIA} &= K^0_H(\partial X_9) \\ \mathcal{Q}^{RR}_{IIB} &= K^1_H(\partial X_9) \,, \end{aligned} \tag{1.3}$$

where $\mathcal{Q}^{RR}$ are the charges defined by measuring the RR fluxes at the boundary. In the geometries we have considered the IIA fluxes at infinity are

$$K^0_H(X_8) = K^1(X_5) \otimes K^1_H(\mathrm{SU}(2)) \cong K^1(X_5) \otimes Z_k \tag{1.4}$$

and are therefore k-torsion. In (1.4) we used that $K^0_H(\mathrm{SU}(2)) = 0, K^1_H(\mathrm{SU}(2)) = Z_k$.

Let us now focus on the group of D0 charges as measured by fluxes at infinity. According to (1.4) this group is Z_k. In this paper we show that this Z_k conservation law can be understood as a the result of a Higgs mechanism. A field with charge k gets a vacuum expectation value and therefore we only have a Z_k conservation law. We will also show that this answer can be misleading in some cases. In those cases there are "singleton" modes that can keep track of k units of charge and therefore the full charge group measured at infinity is really Z. Whether this happens or not depends on the behaviour of the metric and dilaton near the boundary and it is not captured by K-theory which depends only on the topology of the boundary.

It is important to bear in mind that in this paper we are defining the charge group as the possible fluxes that we can measure at infinity. This should be distinguished from the topological classification of D-brane sources (which is sometimes called the group of D-brane charges). This latter group is given by the sets of topologically distinct (in string field theory sense) D-brane configurations. This is given by the K-theory of the whole space (as opposed to the K-theory of the boundary, which classifies the RR fluxes). In fact one can connect the two by saying that [3]

$$\begin{aligned} \mathcal{Q}^D_{IIA} &= K^0_H(\partial X_9)/j(K^0_H(X_9)) \\ \mathcal{Q}^D_{IIB} &= K^1_H(\partial X_9)/j(K^1_H(X_9)) \,, \end{aligned} \tag{1.5}$$

where j is the inclusion map. The essential idea is that this is a form of Gauss' law: Charges are measured by fluxes at infinity. Thus the $K^1_H(X_9)$ charges usually associated with IIA theory are measured by $K^0_H(\partial X_9)$. In accounting for the charges of D-branes we should mod out by those classes which extend smoothly inside since such classes can exist without a D-brane source.

2. IIA spacetime interpretation

In this section we explain how one can understand the Z_k D0 charge group in terms of IIA supergravity. We then explain how the restoration of the U(1) symmetry fits in.

2.1 Higgs mechanism

We will consider spacetimes X_{10} which are fibered over 7-dimensional spacetimes X_7 by S^3, where S^3 carries k units of H-flux. We will perform a Kaluza-Klein reduction along the S^3 and keep only the lowest energy modes. In this section we only keep the most important terms for the physical discussion. Thus, the RR 3-form potential is reduced by

$$C_3 = \chi(x)\omega_3 + c_3 \,, \tag{2.1}$$

where $\chi(x)$ is a scalar in X_7 and c_3 is pulled back from X_7. Similarly, the RR 1-form $C_1 \to c_1$ is assumed to be pulled back from X_7.

As for the other fields in the theory, the H-flux is frozen to be $H = k\xi_3$, and we neglect metric perturbations. This ansatz can be justified in supersymmetric backgrounds such as the throat geometry of the extremal fivebrane since these have no tachyons. In other examples, a more careful analysis is needed. Even if there are tachyons we believe that our discussion captures the main physics.

The RR sector of the IIA supergravity lagrangian reduces to

$$S = \int_{X_7} |d\chi + kc_1|^2 + |dc_1|^2 + |dc_3|^2 + kc_3 dc_3 \,. \tag{2.2}$$

The RR U(1) gauge symmetry acts on χ because it acts on C_3 in ten dimensions $\delta C = D_H\Lambda \equiv (d - H\wedge)\Lambda$. This gauge transformation law shows that χ shifts in the appropriate way so that the order parameter

$$\Phi = e^{i\chi} = \exp\left[i\int_{S^3} C_3\right] \tag{2.3}$$

has charge k.

From (2.2) it is clear that the U(1) RR gauge symmetry is spontaneously broken by the standard Higgs mechanism. Furthermore, since it is broken by the charge k order parameter $\Phi = e^{i\chi}$, the U(1) symmetry is spontaneously broken down to Z_k. Since the dual group of Z_k is again Z_k, the Z_k group of D0 charges is explained by a completely standard physical mechanism.

Remarks:

1. There are three distinct and independent sources of mass terms in the 7 dimensional theory which should not be confused with one another. First, the linear dilaton gives mass to some NS fields. Second there is a gauge invariant mass term for the 3-form C_3 in 7-dimensions giving this field a mass of order $1/k$ see (2.2). This mass term does *not* break the U(1) RR gauge symmetry associated with c_1. Neither does it break the gauge symmetry of the c_3 field. Indeed, note that (2.2) is a sum of two decoupled systems. Third, there is a *Higgs* type lagrangian for the coupling of χ to the RR vector field c_1. This is the term responsible for the symmetry breaking and the consequent torsion charge. In appendix A we will review several general facts about Chern-Simons/BF theories, and in particular show that the Higgs-type couplings are dual to the Chern-Simons-like terms.

2. Several authors have discussed several definitions of charge [13, 14, 15, 16, 17]. In the viewpoint advocated in this section, there is in fact no U(1) charge simply because a charged field has obtained an expectation value, and we cannot define a charge in a spontaneously broken vacuum.

3. In the usual discussion of the Higgs mechanism a potential is responsible for fixing the modulus of the Higgs field. It is possible to make the modulus of Φ in (2.3) dynamical and have a lagrangian in which the U(1) gauge symmetry is realized linearly. In such a lagrangian a potential must fix the expectation value of $|\Phi|$ to one and then this massive field can be integrated out. One would have to perform a more detailed Kaluza-Klein analysis to learn about this potential. For our purposes it is enough to consider the simpler problem without a dynamical $|\Phi|$ which is based on (2.2).

2.2 How to measure torsion charges

One interesting question is whether we can measure torsion charges at long distances. It seems clear that if a charge is conserved in a quantum theory of gravity there should be a way to measure it at long distances, otherwise the charge can fall into a black hole and disappear. In the present case we can measure these charges by measuring Aharonov-Bohm phases in the following way. For simplicity consider the NS-5 brane, but the same could be repeated for the other backgrounds mentioned above. Suppose we want to measure the D0 brane Z_k charge. We consider a D4 brane that is a point on S^3. This implies that in the extra seven dimensions the D4 brane has codimension two. Then we take the D4 brane around the D0 branes and we produce a phase

$$e^{i2\pi n/k}\,, \tag{2.4}$$

where n is defined as the D0 brane charge. Clearly this charge is defined modulo k and furthermore, we can measure it at long distances. This phase comes from the Chern Simons couplings of the RR gauge potentials. More explicitly, in the ten dimensional IIA supergravity lagrangian there is a coupling of the form $\int dC_5 \wedge A \wedge H \sim k \int dC_5 \wedge A$, where C_5 dual to C_3. (In terms of C_3 this coupling comes from the kinetic term which is of the form $(dC_3 + A \wedge H)^2$). At long distances we can neglect the kinetic terms for C_5 and A. In the presence of a D4 brane the equation of motion for C_5 implies that the field A around the fourbrane is of the form $A_\varphi \sim 1/k$ where φ is the angle around the fourbrane. So when we take the D0 brane around the fourbrane we get the phase (2.4) . One can similarly measure D2 brane charges by taking them around other D2 branes oriented in different directions. In principle it should be possible to compute this directly from the one loop open string diagrams. This has indeed been done in [18] for type I torsion charges. It seems that one should be able to measure NS torsion charges via similar Aharonov-Bohm phases. There are several interesting open problems related to these questions. For example, it would be nice to have a simple K-theory formula for such phases.[5]

3. M-theory picture

All the type IIA backgrounds described in this paper have a simple M-theory lift. We just need to add the M-theory circle using the standard formulas for the uplifting.

[5] A natural proposal for such an expression is the following. Consider IIA theory on $X_9 \times R_t$. Suppose a brane produces a torsion flux. (It must be a torsion flux in order to be able to speak of Aharonov-Bohm phases in the first place.) Accordingly, we get an element of $K^0_{tors}(X_8)$ where $X_8 = \partial X_9$. We are going to measure the phase at infinity. Using the exact coefficient sequence we lift the torsion element to an element of $K^{-1}(X_8; \mathrm{U}(1))$. Now, the test brane defines an element of $K^0(X_8)$. There is a natural pairing $K^0(X_8) \times K^{-1}(X_8; \mathrm{U}(1)) \to K^{-1}(X_8; \mathrm{U}(1))$. Thus, to a torsion flux and test brane charge we get an element of $K^{-1}(X_8; \mathrm{U}(1))$. Now let us consider the Aharonov-Bohm phase for transport along a element of $H_1(X_8)$. We lift this to an element of K-homology and consider the pairing $K_1(X) \times K^1(X; \mathrm{U}(1)) \to \mathrm{U}(1)$. While this pairing exists on purely topological grounds it can also be defined in terms of eta invariants of Dirac operators [19], and these likewise enter in discussions of Aharonov-Bohm phases. Thus, it is natural to guess that this is the Aharonov-Bohm phase. We thank E. Witten for a discussion on this matter.

Our conventions for the relation of IIA theory and M theory are the following. 11-dimensional spacetime is a circle bundle with a globally well-defined 1-form

$$\Theta = d\varphi_M + C_{1,\mu} dx^\mu \,. \tag{3.1}$$

We have $0 \leq \varphi_M \leq 2\pi$ so that Θ is normalized to $\int_{S^1} \Theta = 2\pi$ along the fiber. Denote the fieldstrength by $R_2 = dC^{(1)}$. The metric is taken to be

$$ds^2_{(11)} = e^{4\phi/3}\Theta^2 + e^{-2\phi/3} ds^2_{IIA} \tag{3.2}$$

with ϕ independent of the the fiber coordinate. Thus there is a U(1) isometry of the metric. We will decompose the G-field as follows:

$$G^{M-theory} = \pi^*(R_4) + \pi^*(H) \wedge \Theta \,. \tag{3.3}$$

We have written the explicit pullback by $\pi : X_{11} \to X_{10}$ for emphasis, but will henceforth drop it. This defines R_4. Note that $d\Theta = \pi^*(R_2)$ is a basic form so the last term is just part of the definition of R_4. From (3.3) we get the Bianchi identities:

$$\begin{aligned} dR_4 - H \wedge R_2 &= 0 \\ dH &= 0 \,. \end{aligned} \tag{3.4}$$

If the line bundle is trivial we have a globally well-defined 1-form C_1. Then we can write

$$R_4 = dC_3 - H \wedge C_1 \,. \tag{3.5}$$

It follows from (3.3) that in the backgrounds under consideration we have k units of flux of G_4 on $S^1 \times S^3$. The circle size is bounded above and the geometry is non-singular. From this perspective the U(1) D0 symmetry becomes translation along the 11th direction. The metric and the field strength are translation invariant, so it might come as a surprise that translation symmetry is broken. To understand this phenomenon consider an M2 brane whose worldvolume wraps the S^3. Its phase is given by

$$e^{i \int_{S^3} C_3} = e^{ik(\varphi_M - \varphi_M^0)} \,, \tag{3.6}$$

where φ_M^0 is an arbitrary constant.[6] The fact that we cannot define the phase in a U(1) invariant fashion breaks the symmetry from U(1) $\to Z_k$. This effect is very familiar in the context of a two-torus with a magnetic field where the translation

[6] One should be more accurate here. The M-theory 3-form can be viewed as a Cheeger-Simons 3-character. It is a group homomorphism from the group of 3-cycles in 11 dimensions to U(1) such that, if $\Sigma_3 = \partial B_4$ then $\exp[i \int_{\Sigma_3} C_3] = \exp[i \int_{B_4} G_4]$, where G_4 is a closed form with (2π)-integral periods. The formula (3.6) is really the ratio of characters for the 3-cycles $S^3 \times P$ and $S^3 \times P_0$, where P, P_0 are two points on the M-theory circle. In the extremal 5-brane one could also fill in the 3-sphere in which case one would find $e^{ik\varphi_M}$ up to exponential corrections in the radius from the core of the 5-brane.

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group is similarly broken.[7] The fact that such M2 brane instantons break the U(1) symmetry is intimately related to the picture advocated in [20].

Since we have a background with G_4 flux we would be tempted to interpret it in terms of M5 branes. Notice however that the background is non-singular, so it is not obvious where the branes are located. Nevertheless we can think of these backgrounds as being the dual gravity description of a system of M5 branes with a small transverse circle[8] S^1_M. The U(1) symmetry associated with D0 charge corresponds to translations along the circle S^1_M. Since the fivebranes are transverse to this circle, it is clear that we will break the U(1) symmetry. What is a bit surprising is that a Z_k seems preserved, as would be the case if the branes were equally spaced. In general we can say that there is no evidence that the Z_k is broken from the gravity perspective, suggesting that the branes are equally spaced. Furthermore, in the gravity backgrounds we are considering, there are no zero modes corresponding to the motion of individual branes.

In example 2 above we can go even further and make an argument showing that the branes are dynamically restricted to be equally spaced. The argument is as follows. Example 2 can be thought of as arising from IIA NS5 branes wrapped on the S^2 of the small resolution of the conifold giving rise to an effective $d = 4$ $N = 1$ theory in the IR. When the radius of the 11th dimension is large this theory seems to have moduli corresponding to the position of the NS fivebrane on x_M (which together with the expectation value of the dynamical two form field b_2 on the 5-branes on S^2 gives a chiral multiplet). It was shown in [21] that M2 branes stretched between the fivebranes give rise to a superpotential. For large radius of S^1_M, R_M, this superpotential has a minimum when the branes are equally spaced. We conclude therefore that the theory has a massive vacuum with a Z_k symmetry. By holomorphy we expect that this will be the case also for small R_M when the dual gravity description is valid. In summary, for example 2 the Z_k symmetry should be an exact symmetry. In example 1 we do not know of an argument that will tell us that the Z_k symmetry survives all possible non-perturbative effects. In fact example 1 is already perturbatively unstable.

Finally let us discuss the case of the usual extremal fivebranes that preserve 16 supercharges. In this case we can move all the fivebranes independently and there is no reason for expecting a Z_k symmetry. Indeed, in the IIA gravity solution that describes this system there is a strong coupling region. D0 branes can fall into this region and disappear. If the branes are equally spaced, then we expect a Z_k symmetry

[7]One might ask whether in a case where we have flux on S^4 the symmetry is broken. In this case it is possible to define the phase in an SO(5) invariant fashion by choosing a four- manifold inside S^4 whose boundary is the $M2$ brane, as it is done in the familiar case of the SU(2) WZW model, which has the full symmetry of S^3.

[8]The condition of the circle being small is really the condition that the decoupling limit (or nearly decoupling limit) is that of the IIA NS fivebrane, which has an M-theory lift in the IR.

in the full non-perturbative theory. The precise symmetry group depends on what the theory is doing at the IIA strong coupling singularity.

4. D-brane probes

D-brane probes are very useful in assessing the symmetries of a background. Gauge symmetries of the background become global symmetries of the probe theory. So an interesting perspective on the symmetry breaking U(1) $\to Z_k$ is provided by considering a D2-brane probe in the 5-brane background. To be definite, consider a probe in examples 1 or 2 whose worldvolume is along the directions where the IIA metric is trivial. So the probe will be a point on S^3.

The probe theory is a 3-dimensional U(1) gauge theory. As is standard, we dualize the photon by

$$F = dA = *d\varphi_M \, . \tag{4.1}$$

We can interpret the scalar field φ_M as the position of the M2 brane on the 11th circle [22], and identify our global U(1) symmetry as dual to the gauge U(1). Since the gauge group is U(1), and not $\mathbb{R}$, there are instantons for the U(1) gauge field, $dF \sim \sum_i \delta^3(\sigma - P_i)$, which break the global U(1) symmetry [23].

The duality (4.1) is valid in the IR for the probe field theory regardless of whether the radius of the M-theory circle is large or not, i.e. regardless of whether we can use the eleven dimensional description for the whole background. The instantons which correct the 3 dimensional theory are obtained by considering the full theory. They consist of D2 brane world volumes wrapping S^3. The H field on S^3 gives them magnetic charge under F. In the M-theory description, they are $M2$ brane world volumes wrapping S^3 which have an amplitude

$$\sim \exp\left[-T_{M2}\mathrm{vol}(S^3) + i\int_{S^3} C_3^M\right] \, , \tag{4.2}$$

where C_3^M is the M-theory RR 3-form potential.[9] Since $G_4 = dC_3^M = k\omega_3 \wedge d\varphi_M$, the phase of the instanton amplitude (4.2) becomes $e^{ik(\varphi_M - \varphi_M^0)}$. Consequently, the instantons generate terms in the low energy effective action which are proportional to

$$e^{-T_{M2}\mathrm{vol}(S^3)} e^{ik(\varphi_M - \varphi_M^0)} \tag{4.3}$$

demonstrating an explicit breaking U(1) $\to Z_k$.

Remark:

We have focused on the instantons relevant for writing the low energy effective action on the brane. Nevertheless, it is interesting to ask about the effects of instantons

[9] Note that in our examples the amplitude of (4.2) is small only if k/g_s is large, i.e. only if we are in the perturbative IIA regime.

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which simultaneously wrap both the fiducial worlvolume of the probe *together with* the S^3. Such instantons have zero scale size because of a certain "bubbling phenomenon" present for the harmonic map problem in dimensions larger than two. Briefly, suppose we have a map $F : M \to M \times M$ where M is an n-dimensional manifold equipped with metric g and B-field. Consider the action

$$I[F] := \int_M \mathrm{vol}(F^*(E \oplus E)) \,, \tag{4.4}$$

where $E = g + B$. If F has degree $(1,1)$ then for $n > 2$ there is a bubbling instability. Indeed, let $h(x)$ be a test function mapping the ball $\| x^\mu \| \leq 1$ (in some metric, in some coordinate patch) onto M with degree 1. Moreover, suppose that on the boundary h maps to a single point P_0. We can then extend h to the rest of M simply by letting everything outside the ball $\| x^\mu \| \leq 1$ map to P_0. Now we define $h_\lambda(x) := h(x^\mu/\lambda)$ for $\| x \| \leq \lambda$ and h_λ maps everything outside this ball to P_0. It is not difficult to see that the configuration is unstable to shrinking $\lambda \to 0$ when $n > 2$. This leads to the bubbled configuration where $F(x) = (x, f(x))$ with $f(x)$ wrapping once around M at $x = 0$ and $f(x) = $ constant for $x \neq 0$.

5. Restoration of the U(1)

Let us return to the general backgrounds of section 1.1. We have explained how K-theory predicts that the D0 charge group in such backgrounds is Z_k. Equivalently, the U(1) gauge symmetry of the RR 1-form of IIA theory is broken to Z_k. We have explained this in terms of the Higgs mechanism, brane probes, and dynamically generated symmetries in M-theory. We will now explain that these arguments can be wrong. More charitably, we will show that they present a limited point of view.

More precisely, we will explain that, in certain circumstances, the unbroken group should really be regarded as U(1), and not Z_k, and consequently the group of D0 charges is Z, and not Z_k. The apparent contradiction with the K-theoretic formulation is resolved when one understands that the "symmetry restoration" to U(1) involves degrees of freedom ignored in the K-theory analysis.

5.1 RR collective coordinates

One viewpoint on the symmetry restoration is that there is a collective coordinate of the RR fields, which arises since "large RR gauge transformations" which do not vanish at infinity should be considered as global symmetries, and not gauge symmetries.

Recall that $H = k\xi_3$ for some 3-form ξ_3 with integral periods such that $\xi_3 \to \omega_3(1 + \mathcal{O}(1/r))$ at infinity. The equations of motion imply that $d\xi_3 = 0$ and $d(*e^{-2\phi}\xi_3) = 0$. Moreover, we assume that ξ_3 is normalizable, which implies that the fivebrane has finite volume.

Under these conditions there exists a normalizable collective coordinate for the RR fields:

$$
\begin{aligned}
C_3 &= \chi(t)\xi_3 \\
C_1 &= -\frac{1}{k}\dot{\chi}(t)(1-e^{-2(\phi-\phi_\infty)})dt\,.
\end{aligned} \tag{5.1}
$$

Thus, $R_4 = -\dot{\chi}(t)e^{-2\phi}\xi_3 \wedge dt$ satisfies the Bianchi identity $dR_4 - H \wedge R_2 = 0$. The zeromode of $\chi(t)$ provides a solution of the RR equations of motion. The normalizability of the zeromode of $\chi(t)$ is determined by that of ξ_3.

Large global gauge transformations show that $\chi(t)$ is a periodic variable, with period $\chi \sim \chi + 2\pi$. In the quantum mechanical theory, this periodic coordinate has a spectrum of discrete momenta, each unit of momentum carries k units of D0 charge, which together with the previous Z_k form the D0 charge group which is Z. A more detailed analysis of this point will be given in section 6.

Now we can see what was missing in the analysis of the previous sections. First, in the discussion of the Higgs mechanism, in section 2.1 the zeromode of the Goldstone boson for the RR 1-form is normalizable and hence does not induce spontaneous symmetry breaking U(1) $\to Z_k$. Second, in the M2 probe analysis of section 4, the zeromode φ_0^M in (4.3), must be integrated over, so the mode becomes dynamical and the symmetry is unbroken. This mode is like an axion since it appears in the instanton amplitudes (4.3) as a θ parameter.

In the M-theory description we are allowing a motion of the M-fivebranes along the 11th dimension. In fact, the easiest way to get the zero modes (5.1) by uplifting the IIA background and considering an infinitesimal boost $\varphi_M \to \varphi_M + vt$, $t \to t + v\varphi_M$. The 4-form transforms as $G_4 = k\xi_3 \wedge d\varphi_M + vk\xi_3 dt$. The metric becomes

$$
\begin{aligned}
ds_{11}^2 &\sim e^{4\phi/3}(d\varphi_M + vt)^2 + e^{-2\phi/3}\Big[-(dt + vd\varphi_M)^2 + \cdots\Big] \\
&\sim e^{4\phi/3}[d\varphi_M + v(1-e^{-2\phi})dt]^2 + e^{-2\phi/3}\Big[-(dt)^2 + \cdots\Big]
\end{aligned} \tag{5.2}
$$

up to $\mathcal{O}(v^2)$. This fixes C_1 and we can now obtain C_3 from (3.3) to obtain the new RR potentials. These are just those given in (5.1) above. To the extent that we can think of this geometry as a wrapped 5-brane, the 5-brane is wrapped on the 5-manifold X_5 and propagates in time. Therefore, its mass is finite iff vol(X_5) is finite at $r = \infty$. If there are k M5 branes transverse to the M-theory circle with coordinate φ_M, and the branes have a finite mass (e.g., because they wrap a finite volume space) then these five-branes can absorb momentum and move in the φ_M direction.

What is a bit surprising from this point of view is that the quantization of the singleton mode gives us k units of momentum. The reason is that the transformation $\chi \to \chi + 2\pi$ in the IIA variables corresponds to motion of all the branes by $\Delta\varphi_M = 2\pi/k$ along the 11th circle. The fact that this is a symmetry is more evidence that the fivebranes are equally spaced. More precisely, we should consider the wavefunction for the system including the position of all 5-branes $x_1, \ldots, x_k$ around the M-theory circle. We can separate the center of mass degree of freedom $x_{cm} = \frac{1}{k}\sum_i x_i$ and accordingly separate the wavefunction

$$\Psi(x_1, \ldots, x_k) = e^{2\pi i q x_{cm}} \Psi_q^{\mathrm{rel}}(x_i - x_j)\,, \tag{5.3}$$

where we have written a state in an eigenstate of total momentum around the M-theory circle of momentum $q \in Z$. The relative wavefunction only depends on $q \bmod k$, through the phases that appear when we change one of the coordinates $x_i \to x_i + 2\pi$. Therefore, when we change $q \to q+k$ there is no change in the relative wavefunction and this mode is captured by a simple collective coordinate as above. On the other hand if we change q by another amount we need more information about the system to determine the relative wavefunction.

5.2 Some U-dual descriptions: KK-monopoles and H-monopoles

An interesting perspective on the phenomena discussed in this paper is provided by the story of unwinding strings in the presence of Kaluza-Klein monopoles [24]. This example also illustrates the need for interpreting (1.3) with care.

Consider first type IIB on $R^{1,4} \times X_5$ with flat metric on $R^{1,4}$ and zero H-field. Then $\partial X_9 = S^3 \times X_5$. The reader may set $X_5 = T^5$ without much loss of generality. Let us consider the RR fluxes in such a background. For IIB we should compute the flux group at infinity

$$K^1(\partial X_9) = K^1(S^3) \otimes K^0(X_5) \;\oplus\; K^0(S^3) \otimes K^1(X_5)\,. \tag{5.4}$$

We are interested in fluxes associated to D1 branes in $R^{1,4}$. These are measured (in cohomology) by a degree seven class on the boundary (dual to H_3^{RR} in cohomology) which is of degree five on X_5 and of degree two on S^3. There is no such class in (5.4) . This is not at all surprising since there are no one cycles in R^4 where the D1 branes can be wrapped.

Now let us consider $R^{1,4} \times X_5$ but with a KK monopole, or Taub-NUT metric on R^4, and with zero H-field. Since the metric is smooth and since we can take the string coupling to be arbitrarily weak, the K-theory analysis is valid. The topology of this space is the same as in the example above so that the RR fluxes are again (5.4) . In particular, if we view the sphere S^3 at infinity as an S^1 Hopf fibration over S^2, (5.4) predicts there is no charge associated with the winding number of D1 strings around the fiber coordinate. Fluxes associated with such strings would be degree 7 classes associated with elements of $K^0(S^3) \otimes K^1(X_5)$ which are degree two in $K^0(S^3)$, but there are no such classes.

In fact, there *is* a nontrivial RR charge associated with the winding of D1 strings around the Hopf fiber at infinity. The apparent contradiction with K-theory is resolved by noting that the analysis leading to (1.3), (5.4) neglects the collective coordinate degrees of freedom in the RR potentials explained in section 5.1. The collective coordinate for the KK monopole and the fact that it carries RR charge was discussed in [24][10]. The collective coordinate of the KK monopole comes from $B_2^{RR} = \alpha(t)\Omega_2$ where Ω_2 is a normalizable harmonic 2-form on TN [25]. This collective coordinate carries D1 string charge and the charge measured at infinity is indeed Z, signaling an unbroken U(1) gauge group. Again, it is important that X_5 is compact, otherwise this collective coordinate would not be normalizable. In conclusion, it seems that to determine the physically relevant group of charges, as measured by RR fluxes at infinity, one needs more information than just the topology of the space. As we explained above, flat $R^{1,4}$ and the KK monopole have the same topology, but we expect different physical charges at infinity. It would be nice to understand how to correct (1.3) in a general background, so that it gives the physically appropriate answer.

As discussed in [24] it is also quite interesting to consider the T-dual description of the above phenomenon. Now we have type-IIA theory on an H-monopole. A wound D1 string becomes a D0-brane, in the presence of an NS5 brane. The transverse space is $R^3 \times S^1$. At long distances $H \sim \omega_2 \wedge d\theta$ where ω_2 is the unit volume form on the sphere $S^2 \subset R^3$. Therefore, the D0 can end on a D2-instanton wrapping $S^2 \times S^1$. Should we conclude that D0 brane charge, as measured by the corresponding RR flux, will be undefined? As we have seen in section 5, the answer is "no": There is a mode of the IIA RR potential $C_3 \sim \chi\omega_2 \wedge d\theta$, for which our standard story applies: χ is the Goldstone boson eaten by the RR 1-form C_1. Using the explicit form of the metric one checks that $\omega_2 \wedge d\theta$ is normalizable in the weak coupling region of the H-monopole.[11] This closes the circle of ideas.

In the above discussion we pointed out that the RR fluxes are not properly classified by (1.3). On the other hand we have no complaints against (1.5) as a group of D-brane sources. In our case (1.5) together with (5.4) gives us the source charge group $\mathcal{Q}^D \cong K^1(S^3) \otimes K^0(X_5) \cong K^0(X_5)$. This again vanishes for D1 branes wrapping the fiber of the Hopf fibration, but this is not surprising since they can shrink to nothing. So there is no problem in the interpretation of (1.5) as the group of topologically distinct sets of D-branes that we can have in a given background.

The above discussion has been carried out for a singly charged KK monopole. The generalization to charge k monopoles is rather interesting. Let us therefore consider a smooth Taub-NUT space TN_k corresponding to k KK monopoles. The boundary at infinity is topologically a Lens space S^3/Z_k, and metrically the fiber of $S^3/Z_k \to S^2$ is asymptotically of constant radius. There appears to be a conserved

[10] In [24] fundamental strings were considered, here we want to consider D-strings. The discussion is exactly the same after exchanging $B_2^{NS} \to B_2^{RR}$

[11] The above form is non-normalizable in the strong coupling region of an extremal H-monopole. In the philosophy of this paper we should consider a smooth background with this strong coupling singularity removed. Then the mode will be normalizable.

Z_k winding charge since two strings winding n and $n + Nk$ times around the fiber can be smoothly deformed to one another at infinity. Moreover, for the resolved TN space the fundamental group is $\pi_1(TN_k) = 0$, suggesting that there is no winding charge at all. Note that the unwinding of k strings can be done far away from the cores of the KK monopoles, but in order to unwind a smaller number of strings we need to go to the cores of the KK monopoles.

Let us now consider the RR charge in the K-theoretic description. The K-theory of 3-dimensional Lens spaces is easily computed [26]:

$$\begin{aligned} K^0(S^3/Z_k) &= Z + Z_k \\ K^1(S^3/Z_k) &= Z\,. \end{aligned} \tag{5.5}$$

The nontrivial classes in K^0 can be represented by the flat line bundles associated to the unitary representations of $\pi_1(S^3/Z_k) = Z_k$. These line bundles extend to the full space TN_k as tautological line bundles in the hyperkahler quotient construction. Thus, the group of RR fluxes at infinity is

$$(Z + Z_k) \otimes K^1(X_5) \;\oplus\; K^0(X_5)\,. \tag{5.6}$$

The factor $Z_k \otimes K^1(X_5)$ is the group of fluxes associated to strings winding around the nontrivial elements of $\pi_1(S^3/Z_k)$. So we would conclude from this that the RR fluxes associated to D1 branes wrapping the fiber is Z_k-valued. As we explained above the collective coordinate degrees of freedom imply that this flux is actually Z-valued. In fact, in this case there are many normalizable harmonic two forms Ω_2, which lead to RR zero modes. There is essentially one for each KK monopole.

It is interesting also to compute the group of D-brane source charges by modding out by the image of j in (1.5). In this case the image of j is nontrivial since, as we mentioned above, the Z_k fluxes can be extended to the interior. This implies that the group of source charges is again $K^0(X_5)$, so that there is no source-charge coming from D1 branes wrapping the fiber. This is consistent with what we said above, since a D1 brane wrapping the fiber can shrink to nothing.

The T-dual of this charge k situation brings us back to the D0 charge group. The unwinding of the D1 strings at infinity is T-dual to the decay of k D0 branes due to a D2 brane instanton. From the K-theoretic viewpoint we compute (e.g. via the AHSS)

$$\begin{aligned} K^0_H(S^2 \times S^1) &= Z \\ K^1_H(S^2 \times S^1) &= Z + Z_k \end{aligned} \tag{5.7}$$

with $H = k\omega_2 \wedge d\theta$. The Z_k summand in the second line corresponds to D0 charge, and the Z summand in the second line corresponds to D2 branes wrapping S^2. In this T-dual picture we see the Z_k group in the weakly coupled region, but the fact that it can be broken to nothing depends on the behaviour of the theory in the strong coupling region, where K-theory is not valid. However, by duality we know that in the strong coupling region we should think in terms of M fivebranes and that the Z_k group is broken if we put the fivebranes at generic positions in the 11 dimensional circle. It is also broken if we separate the 5-branes in the transverse three dimensional space. Separating them in the three dimensional space is U-dual to resolving the the singular charge k KK monopole into the smooth Taub-NUT space described above.

Finally, we note that one could combine the stories and consider k_2 H-monopoles together with k_1 KK monopoles. The relevant K-theory group is computed from

$$\begin{aligned} K_H^0(S^3/Z_{k_1}) &= Z_{k_1} \\ K_H^1(S^3/Z_{k_1}) &= Z_{k_2}\,. \end{aligned} \tag{5.8}$$

Here $k_1k_2 = k \neq 0$ and $H = k_2x_3$ is normalized so that $\int_{S^3/Z_{k_1}} x_3 = 1$. Note that (5.8) is nicely compatible with T-duality.

6. A detailed analysis of the singleton mode and U(1) symmetry restoration

In this section we focus on the background given in example 1 and we compactify the five directions along the fivebrane. We dimensionally reduce to the r, t directions and retain only the RR potentials χ and C_1 (and we rename the latter potential to be A in this section). The resulting lagrangian is

$$S = \frac{1}{4\pi}\int dt \int_{r_0}^{\infty} dr \left[\beta(r)(\dot{\chi} + kA_0)^2 - \gamma(r)(\chi' + kA_1)^2 + \frac{1}{\alpha(r)}F_{0r}^2\right], \tag{6.1}$$

where

$$\begin{aligned} \frac{1}{\alpha} &= \frac{v_5}{(2\pi)^6\alpha'^3}2\pi^2r^3\left(1+\frac{Q_5}{r^2}\right)\left(1-\frac{r_0^2}{r^2}\right) \\ \beta &= \frac{v_5}{(2\pi)^2\alpha'}\frac{1}{2\pi^2r^3\,(1+Q_5/r^2)} \\ \gamma &= \frac{v_5}{(2\pi)^2\alpha'}\frac{(1-r_0^2/r^2)}{2\pi^2r^3\,(1+Q_5/r^2)^2}\,, \end{aligned} \tag{6.2}$$

where v_5 is the volume of the directions along the fivebrane at large r, and $(\alpha' k)^2 = Q_5(Q_5 + r_0^2)$.

Before proceeding we do a duality transformation in χ to the variable $\tilde{\chi}$ so that the lagrangian becomes

$$S = \frac{1}{2\pi}\int dt \int_{r_0}^{\infty} dr \left[\frac{1}{2\alpha}F_{r0}^2 + \frac{1}{2\gamma}(\dot{\tilde{\chi}})^2 - \frac{1}{2\beta}(\tilde{\chi}')^2 + k\tilde{\chi}F_{0r}\right] + \frac{k}{2\pi}\int dt A_0\tilde{\chi}(r=r_0)\,, \tag{6.3}$$

where $\tilde{\chi}$ is defined as $C_5 = \tilde{\chi}\xi_{v_5}$ where ξ_{v_5} is fiveform along the volume of the fivebrane normalized so that its integral is one. Both χ and $\tilde{\chi}$ are periodic with periods 2π. C_5 is the field dual to C_3 in ten dimensions, $dC_5 \sim *(dC_3 + HA)$. The last term in (6.3) is a boundary term necessary in order to have the appropriate boundary conditions at the origin ($r = r_0$). More precisely, we want to have boundary conditions at the origin such that A_0 is free, therefore we need the boundary term to cancel a boundary contribution in the variation of the action. We also impose that $\tilde{\chi}(r = r_0)$ =constant. When the time direction is a circle, large gauge transformations determine this constant

$$\tilde{\chi}(r = r_0) = 2\pi n/k\,. \tag{6.4}$$

n can be interpreted as the number of D0 branes at the origin. For simplicity we will not put any D0 branes at the origin. In this case the boundary condition is $\tilde{\chi} \in 2\pi Z$. With these boundary conditions the full ten dimensional solution is non-singular at the origin. The reason for this is that the five-form must be smoothly extendible to the entire geometry. In examples 1 and 3 of section 1.1 there is a circle along the brane worldvolume which is contractible to a point at $r = r_0$. In example 2 the there is a 2-sphere along the worldvolume contractible to a point.

We add to the system a number of Wilson line observables which are D0 worldlines of charges q_i which we take for simplicity to be purely temporal. Then the A_0 equation of motion is

$$\partial_r\left(\frac{1}{\alpha}F_{0r}\right) + k\partial_r\tilde{\chi} + 2\pi\sum q_i\delta(r - r_i) = 0\,, \tag{6.5}$$

where q_i are integers.

We *define* the D0 brane charge to be the RR flux measured at infinity:

$$Q \equiv \frac{1}{2\pi\alpha}F_{r0}|_{r=+\infty}\,. \tag{6.6}$$

By integrating (6.5) we conclude that the total charge is

$$Q = k\frac{(\tilde{\chi}(\infty) - \tilde{\chi}(0))}{2\pi} + \sum_i q_i\,. \tag{6.7}$$

The value of $\tilde{\chi}(\infty)$ is a theta angle and it gives a fractional value to the D0 brane charge of a fivebrane by the usual Witten effect. Without much loss of generality we can set it to zero, since it is just an overall shift in the charge. Note that even though $\tilde{\chi}$ is only defined modulo 2π, the difference $\tilde{\chi}(\infty) - \tilde{\chi}(0)$ is a well-defined real number if $\tilde{\chi}$ is continuous. We then identify $k(\tilde{\chi}(\infty) - \tilde{\chi}(0))/(2\pi)$ as the contribution to the charge from the singleton.

In order to gain a bit more insight notice that we could think of spacetime as divided into two regions, the region where the photon is massive, which is the throat region and the region where it is massless which is the asymptotic region far away from the brane.

In order to get some intuition on the behaviour of the system let us understand the dynamics in the massive region. So let us set, α, β, γ to constants $\beta = 1$, $\gamma = \gamma_0$ and $\alpha = \alpha_0$. Moreover, we introduce a spatial variable ρ, with $-\infty < \rho < \infty$. Let us find the fields produced by a Wilson line of charge q in the time direction inserted at $\rho = 0$. The equation of motion for $\tilde{\chi}$, for a time independent configuration, is

$$\partial_\rho^2 \tilde{\chi} + k F_{0\rho} = 0\,. \tag{6.8}$$

We can solve (6.5) by setting

$$\frac{1}{\alpha_0} F_{0\rho} + k\tilde{\chi} + 2\pi q \theta(\rho) = 0\,. \tag{6.9}$$

The solution is

$$\begin{aligned} \tilde{\chi} &= 2\pi \frac{q}{k}\left(\frac{1}{2}e^{-\kappa\rho} - 1\right), & \frac{1}{\alpha_0}F_{0\rho} &= -2\pi q \frac{1}{2} e^{-\kappa\rho} & \rho > 0 \\ \tilde{\chi} &= -2\pi \frac{q}{k}\frac{1}{2} e^{-\kappa|\rho|}\,, & \frac{1}{\alpha_0}F_{0\rho} &= 2\pi q \frac{1}{2} e^{-\kappa|\rho|} & \rho < 0\,, \end{aligned} \tag{6.10}$$

where $\kappa = k\sqrt{\alpha_0}$.

So we see that as we cross the Wilson line from $\rho < 0$ to $\rho > 0$ the value of $\tilde{\chi}$ jumps by $-2\pi q/k$. We can obtain the long distance version of this result by considering the long distance version of the lagrangian (6.3

$$S \approx \frac{1}{2\pi}\int k\tilde{\chi}F \tag{6.11}$$

which is a "BF" theory in two dimensions. (See appendix A for some relevant facts about such theories.)

In conclusion, we find that even though the electric field produced by a point charge decays exponentially, the $\tilde{\chi}$ field "remembers" how much charge there was. Since $\tilde{\chi}$ has period 2π the charge is defined modulo k in the massive region.

In order to get some insight for the singleton, let us consider a simplified system with a boundary at $\rho = 0$. We need some boundary conditions at $\rho = 0$. We impose boundary conditions $A_0(\rho = 0) = \tilde{\chi}(\rho = 0) = 0$. The first boundary condition implies that the U(1) gauge transformation parameter is independent of time at the boundary. The constant part is a global symmetry, i.e. we only divide the path integral by gauge transformations where the gauge parameter goes to zero at the boundary. The condition that $\tilde{\chi} = 0$ will be more fully justified later. Time independent solutions of the resulting equations are of the form

$$\tilde{\chi} = \tilde{\chi}(-\infty)(-e^{-\kappa|\rho|} + 1)\,, \qquad \frac{1}{\alpha_0}F_{0\rho} = k\tilde{\chi}(-\infty)e^{k\rho} \qquad \rho < 0\,, \tag{6.12}$$

where $\tilde{\chi}(-\infty)$ is the value of $\tilde{\chi}$ as $\rho \to -\infty$. This is the singleton mode that carries the charge. The total charge is given by (6.6) evaluated at the boundary, and this can be read off from (6.12). If there are no other charges in the interior, we should use that $\tilde{\chi}(-\infty)/(2\pi)$ is an integer and then we get that the singleton carries charge k. More precisely, in situations where the fivebrane geometry is cut off at some finite value of $r = r_0$ we argued above that it should be an integer.

This simplified model appears naturally in the theory of the extremal fivebrane as follows. In that case, $\beta = \alpha/\alpha_0$ with $\alpha_0 = (2\pi)^8\alpha'^4/v_5^2$. After defining a new variable ξ through the equation $d\xi = \beta dr$ we find that we get a theory like (6.3) with $\beta \to 1$, $\gamma \to \gamma^2$, $\alpha \to \alpha_0$. Explicitly,

$$\xi - \xi_0 = \frac{v_5}{(2\pi)^4\alpha' Q_5}\log\frac{r^2}{r^2+Q_5} \tag{6.13}$$

and we choose ξ_0 so that the physical range of ξ is $-\infty < \xi < 0$. The region of $\xi \sim 0$ corresponds to the asymptotically flat region far from the fivebrane. The fact that γ is nonzero does not modify any of the time independent equations we considered above. It does, however, have an important consequence. First it implies that $\tilde{\chi}$ is massless in the region $\xi \sim 0$. Second it implies that the only reasonable boundary condition we can impose on $\tilde{\chi}$ at $\xi = 0$ is a Dirichlet boundary condition. As we explained above this constant value of $\tilde{\chi}(\xi = 0) = \tilde{\chi}(r = \infty)$ is like a theta angle in the full theory. Setting $\tilde{\chi} = 0$ at $\xi = 0$ we can compute the energy contained in the singleton mode by inserting (6.12) with $\tilde{\chi}(-\infty) = 2\pi n$ into the hamiltonian. We find that the energy is

$$E = \frac{1}{2}kn^2 2\pi\sqrt{\alpha_0} = \frac{1}{2}kn^2\frac{M_0^2}{M_5}\,, \tag{6.14}$$

where M_0 and M_5 are the masses of one D0 and one NS5 brane. This is, of course, the expected answer based on BPS bounds and it can be thought of as the energy of k M5 branes with nk units of momentum in the 11th dimensional circle, in perfect accord with the M-theoretic interpretation of the singleton degree of freedom of section 5.1.

So- we can at last answer the question: "What happens when k D0 branes disappear?" Since $\tilde{\chi}$ is a periodic variable we have no physical effect if along some trajectory $\tilde{\chi}$ shifts by 2π. Let us call such a configuration a "Dirac string." If we have k coincident D0 branes, we also have this shift by 2π if we are far away from the D0 trajectories. This suggests that k D0-brane lines could be replaced by a Dirac string, as in figure 1. In the low energy theory (6.11) both are equivalent. In the original lagrangian (6.3), they are not. In fact k D0-brane lines cannot terminate due to current conservation, which in turn follows from gauge invariance. This problem can be solved if we add a new object to the theory which is the baryon vertex [27], where k D0 lines can end. This object acts as a magnetic source for $\tilde{\chi}$; it implies the equation

$$dd\tilde{\chi} = 2\pi\delta^2(x)\,, \tag{6.15}$$

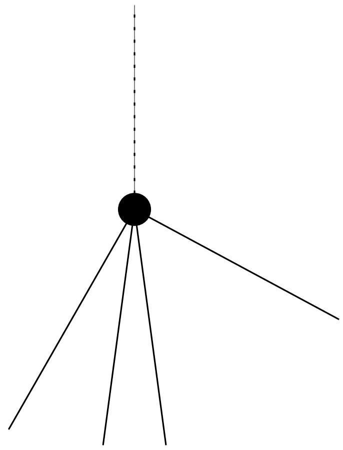

Figure 1: Here we see k D0 lines ending on a baryon vertex. A "Dirac string" for the $\tilde{\chi}$ field, indicated by the dotted line, emanates from the baryon vertex.

where $x = (r, t)$. Note that the winding of $\tilde{\chi}$ around the point where it is inserted is precisely the periodicity of $\tilde{\chi}$. Equation (6.15) also makes sure that the current conservation condition is microscopically obeyed (so that we preserve gauge invariance). The simplest way to understand this is to go back to the original variable χ. If k Wilson lines are ending at the point $x = 0$ there is a term of the form $e^{ik\epsilon(x=0)}$ when we perform a gauge transformation $C_1 \to C_1 + d\epsilon$. This is cancelled if the baryon vertex couples to χ as

$$e^{i\chi(x=0)} \tag{6.16}$$

so as to make the whole configuration gauge invariant. (Recall that the covariant derivative is $d\chi + kc_1$.) Then the equation of motion of χ will be of the form $\nabla^2\chi = 2\pi\delta^2(x)$ which maps under the duality to (6.15).[12] We can also see from the full string theory that the baryon vertex couples to χ as in (6.16). In this case the baryon vertex is a D2 brane worldvolume wrapping the S^3. Since χ is related to the three form on S^3 as in (2.1) the coupling of the three form to the D2 brane translates into (6.16).

In conclusion, the singleton mode is responsible for the conservation of the U(1) charge.

Remarks:

1. It is crucial, for the U(1) to be restored, that the worldvolume of the fivebrane is finite, otherwise the 3-form would not be normalizable.

2. The presence of the singleton mode is also related to the fact that Chern Simons theories have physical propagating modes on the boundary when spacetime has

[12]The laplacian is not that of flat space but involves the functions β, γ when these are not 1.

a boundary and we impose local boundary conditions for the gauge fields. This is the relation to the general theory outlined in appendix A.

7. U-Duality in $AdS_3 \times S^3 \times T^4$

First let us start with a flat space compactification of IIB string theory on $R^6 \times T^4$. The charges of particle like excitations in six dimensions form a lattice $\mathcal{P} \cong Z^{16}$. There are 8 NS charges and 8 RR charges. Similarly there is a lattice of string like charges in six dimensions $\mathcal{S} \cong Z^{10} \cong II^{5,5}$. As representations of $\mathrm{SO}(5,5;Z)$ the particles are spinors and the strings are a vector. When we have a BPS string carrying some charge we can take its near horizon limit, which will generically be $AdS_3 \times S^3 \times T^4$. This system has been much studied. A partial list of relevant references includes [34, 29, 28], [33]–[31].

A choice of near-horizon limit is a choice of string charge $\mathcal{S}$. We regard the **10** as a symmetric bispinor

$$(\mathbf{16} \otimes \mathbf{16})_{\mathrm{symm}} \supset \mathbf{10} \oplus \cdots \tag{7.1}$$

and denote it by $\mathcal{S}_{\alpha\beta} = \mathcal{S}_{\beta\alpha}$. The gamma matrices acting on the spinor representation Z^{16} are integral, so (7.1) can holds for representations of $\mathrm{SO}(5,5;Z)$.

On AdS_3 we will have 16 vector fields corresponding to the sixteen vector fields that we had in six dimensions. These vector fields have a Chern-Simons coupling given by

$$S = \mathcal{S}_{\alpha\beta} \int A^\alpha dA^\beta \,. \tag{7.2}$$

The gauge group for the Chern-Simons fields is $\mathrm{U}(1)^{16}$. Singular gauge transformations such as those described in appendix A induce nontrivial Wilson loops. The group of charges for Wilson loops may be elegantly expressed as follows. From (7.1) we see that a choice of string charge $Q_{\alpha\beta}$ determines a map $\Lambda_Q : \mathcal{P} \to (\mathcal{P})^*$ by $p^\alpha \to g^{\alpha\beta} Q_{\beta\gamma} p^\gamma$ (where $g_{\alpha\beta}$ is the lattice metric). The group of charges is

$$(\mathcal{P})/\Lambda_Q(\mathcal{P}) \,. \tag{7.3}$$

This is the finite group $(Z_{\frac{1}{2}\mathcal{S}^2})^8$ and carries a signature $(4,4)$ quadratic form. In general there is no invariant meaning to the level of the WZW model, and one obtains different levels by going to different asymptotic regions. The only invariant combination is $\mathcal{S}^2$. One particular asymptotic region makes the nature of the group particularly transparent. Consider Q_5 NS5 branes and Q_1 parallel fundamental strings. Thus, in some basis the string charge is $\mathcal{S} = (Q_1, Q_5, 0, \ldots)$. We assume this is a primitive vector, or there are other complications. Thus Q_1, Q_5 are relatively prime. Then we can interpret the charge group as follows:[13]

[13]We thank E. Witten for a useful discussion on this subject.

1. The 4 D1 charges are broken to Z_{Q_5} by euclidean D3's wrapping $S^3 \times T^1$.
2. The 4 D3 charges are broken to Z_{Q_5} by euclidean D5's wrapping $S^3 \times T^3$.
3. The 4 F1 charges are broken to Z_{Q_1} by KK monopoles with $T^1 \subset T^4$ as the Hopf circle.
4. The 4 momentum charges are broken to Z_{Q_1} by NS5-branes wrapping $S^3 \times T^3$, with T^3 orthogonal to the circle carrying the momentum.

The above gives a presentation of the finite group (7.3) as $(Z_{Q_1})^8 \oplus (Z_{Q_5})^8 \cong (Z_{Q_1 Q_5})^8$. Nevertheless, the true charge group is still Z^{16}, by the considerations above. Note that the finite group $(Z_{Q_5})^8$ is the one deduced from $K_H(\partial X_9)$. This example should provide a useful test case for understanding the proper formulation of the classification of fluxes together with charges.

8. Discussion

In some discussions of charges of D-branes the issue of torsion charges takes on an aura of mystery, especially in the K-theoretic context. We would like to emphasize that there is nothing terribly mysterious or exotic about torsion charges. They are present in ordinary laboratory physics in, for example, superconductivity. We have seen that the mechanism by which such torsion charges arise in 5-brane backgrounds is simply through Higgs condensation of a charged field which does not carry the fundamental unit of charge. We can measure these torsion charges by measuring Aharonov-Bohm phases at infinity.

We further pointed out that the physical charge measured at infinity seems to be different from the expression naively deduced from the K-theory of the boundary of the space. The reason is that the definition of the charge far away depends on the *metric* and not just on the topology of the space. This is clearly illustrated by the KK monopole example. RR charges as measured by RR fields at infinity can be carried by RR zero modes as well as by D-branes. We believe this will be an important point in answering the open problem mentioned in the introduction.

Many interesting issues remain open to future investigation. It would be nice to have a clear and systematic criterion for deciding when the U(1) symmetries are, or are not, restored. In example 2 of section 1.1 we believe the U(1) symmetry is *not* restored, since the relevant harmonic three form turns out to be non-normalizable. The D0 charge group is then Z_k and not Z.

The issues we have discussed are intimately related to the topological classification of branes together with fluxes, a subject which has not yet been adequately understood. Finally, the issues we have discussed are related to the relation between U-duality and K-theory. The relation of K-theory to D-brane instantons described in [20] suggests a formulation of an SL$(2, Z)$ invariant version of the AHSS which is relevant to that problem, but we will leave a detailed description of this to another occasion.

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A. Some comments on higher-dimensional Chern Simons/BF theories, and their role in supergravity

At several points in this paper we have relied on Chern-Simons and BF type theories, which are generalizations of the basic paradigm of three-dimensional Chern-Simons theory [35]. Here we collect a few relevant remarks on these theories useful for following the main text. Much of what we say is well-known to those who know it well, and can be found in an extensive literature.

A.1 Duality of Chern Simons and Higgs lagrangians

In equation (2.2) we encountered the two types of systems we would like to study. These lagrangians are easily generalized, and we will study the more general systems.

Let us begin with the general Higgs-type Lagrangion for p-form gauge potentials in D-dimensions. The minkowskian action is

$$\exp\left[-i\int_{X_D}\frac{1}{2}\kappa_1(d\lambda_p+kA_{p+1})\wedge *(d\lambda_p+kA_{p+1})+\frac{1}{2}\kappa_2 dA_{p+1}\wedge *dA_{p+1}\right], \quad \text{(A.1)}$$

where κ_1, κ_2 are some positive constants. The euclidean theory has the same action without the overall factor of i.

Consider first the local physics of (A.1). The classical equations of motion of the theory (A.1) are:

$$\begin{aligned} d[*(d\lambda_p+kA_{p+1})] &= 0 \\ d(*dA_{p+1})+(-1)^p\frac{k\kappa_1}{\kappa_2}*(d\lambda_p+kA_{p+1}) &= 0\,. \end{aligned} \quad \text{(A.2)}$$

The local gauge symmetry includes $A_{p+1} \to A_{p+1} + d\Lambda_p, \lambda_p \to \lambda_p - k\Lambda_p$. After a gauge transformation removing λ_p we recognize the formulae of a massive field with mass-squared $k^2\kappa_1/\kappa_2$. In Minkowski space, wavefunctions are in the induced representation from the antisymmetric tensor of rank $(D-p-2)$ (equivalently, rank $p+1$) of $\mathrm{SO}(D-1)$.

We will now dualize λ_p to λ_{D-p-2} to obtain an equivalent formulation (A.1) in terms of BF theory, but before doing so we should discuss the global nature of the fields.

The global nature of the fields in this theory is of the utmost importance to its proper interpretation.[14] Although we write gauge potentials A_{p+1} and λ_p, when X_D is topologically nontrivial we wish to allow field configurations where they are not globally well-defined. Thus we allow the fieldstrengths $F_{p+2} = dA_{p+1}$ and $f_{p+1} = d\lambda_p$ to be closed forms with nonzero periods. These periods are 2π times an integer, and any integer can appear. Thus the integral constant k appearing in (A.1) is meaningful.[15]

Global aspects are also crucial in the choice of the gauge group. When X_D is compact, the choice most suitable to physics appears to be to allow large gauge transformations $A_{p+1} \to A_{p+1} + \zeta_{p+1}$ where ζ_{p+1} is any closed form with (2π)-integral periods. We will denote the space of such forms by $Z^{p+1}_{2\pi Z}(X_D)$. Similarly, we take as gauge symmetry $\lambda_p \sim \lambda_p + Z^p_{2\pi Z}(X_D)$. It is for this reason that we cannot simply shift away λ_p by a gauge transformation in (A.1).

We are now ready to dualize λ_p to λ_{D-p-2}. At the classical level we can dualize λ_p into λ_{D-p-2} by interchanging equations of motion with Bianchi identities. The equations of motion are solved for

$$\begin{aligned} *(d\lambda_p + kA_{p+1}) &= -\frac{1}{2\pi\kappa_1} d\lambda_{D-p-2} \\ dA_{p+1} &= (-1)^p \frac{k}{2\pi\kappa_2}\lambda_{D-p-2} + \zeta_{D-p-2}\,, \end{aligned} \tag{A.3}$$

where ζ_{D-p-2} is a closed form (this is the constant q of section 6).

We perform the dualization quantum mechanically by following a standard procedure: We introduce an auxiliary $(p+1)$-form B_{p+1} with no restriction that it be closed or have integral periods, together with a new gauge potential λ_{D-p-2} whose fieldstrength has (2π)-integral periods, and consider the action

$$\begin{aligned} \exp\Bigg[-i\int_{X_D}\Big\{\frac{1}{2}\kappa_1(B_{p+1}+kA_{p+1})\wedge *(B_{p+1}+kA_{p+1}) + \frac{1}{2}\kappa_2 dA_{p+1}\wedge *dA_{p+1}\Big\} - \\ -\frac{i}{2\pi}\int_{X_D} B_{p+1}d\lambda_{D-p-2}\Bigg]\,. \end{aligned} \tag{A.4}$$

[14] At this point the reader might wish to rotate to euclidean space and take X_D to be compact.

[15] Some readers will therefore declare that our fields should really be considered to be Cheeger-Simons differential characters [36]. Useful expositions in the physics literature can be found in [37, 38, 39].

Integrating out λ_{D-p-2} forces B_{p+1} to be closed with (2π)-integral periods, and we recover the theory (A.1) together with its solitonic sectors. On the other hand, we could also shift B_{p+1} by kA_{p+1} and perform the gaussian integral on B_{p+1} to obtain the theory

$$\exp\left[-i\int_{X_D}\left\{\frac{1}{8\pi^2\kappa_1}d\lambda_{D-p-2}\wedge *d\lambda_{D-p-2}+\frac{1}{2}\kappa_2 dA_{p+1}\wedge *dA_{p+1}\right\}- -\frac{ik}{2\pi}\int_{X_D}A_{p+1}d\lambda_{D-p-2}\right]. \tag{A.5}$$

The new theory has a topological BF-type coupling. In the euclidean signature the first term is real but the BF coupling remains imaginary. In both theories (A.1) and (A.5) the number of degrees of freedom is the same, it is that of a massive A_{p+1} form. In this sense the Higgs mechanism and Chern-Simons couplings are dual to each other.

An important special case of the above theory occurs in odd dimensions $D = 2n+1$ with $p+1 = n$. In this case we get a Chern-Simons selfcoupling and further discussion is required.

In the present paper the most important examples are

1. $D = 7, p = 0$. The Higgs system is replaced by the BF system with λ_5, A_1.
2. $D = 7, p = 2$. This is the self-dual 3-form.
3. $D = 2, p = 0$. This is the 1+1 dimensional theory analyzed in detail in section 6.

A.2 Topological theory in the long-distance limit

The long-distance/low-energy dynamics of the theory (A.1) is dominated by the BF coupling in the formulation (A.5). This is most easily seen by simply noting that in (A.5) the BF coupling has only one derivative.[16] Thus we obtain a topological field theory of BF type:

$$\exp\left[-\frac{ik}{2\pi}\int_{X_D}A_{p+1}dA_{D-p-2}\right], \tag{A.6}$$

where we have renamed $\lambda_{D-p-2}\to A_{D-p-2}$ in this subsection.

Another example of this phenomenon was studied in detail by Witten in [40] in the context of the AdS/CFT correspondence on $AdS_5\times S^5$. In that case, the large N limit jutified the condensation of $\int_{S^5}G_5 = N$ leading to a 5-dimensional TFT of BF type. While the mathematics is rather similar to what we are discussing in this

[16] At this point some readers might get confused. Usually, in justifying the dominance of topological terms in an action one introduces a family of metrics $g_{\mu\nu} = t^2 g^{(0)}_{\mu\nu}$ and takes a $t\to\infty$ limit. In our case we must also scale the potential A_{p+1}, or, better, its gauge coupling κ_2.

paper, the physics is slightly different, since we obtain a topological field theory by freezing a Neveu-Schwarz sector field.

The topological field theory (A.6) is more subtle than might at first appear. Once again, it is absolutely crucial to specify the global nature of the fields and their gauge symmetries. We allow the on-shell fieldstrengths to be closed forms with (2π)-integral periods. What should we take to be the gauge group? Let B^p be the space of exact p-forms and Z^p the space of closed p-forms. There are three natural choices of gauge group for the field A_i:

1. $A_i \sim A_i + B^i(X_D)$.
2. $A_i \sim A_i + Z^i_{2\pi Z}(X_D)$.
3. $A_i \sim A_i + Z^i(X_D)$.

Accordingly, there are *six* theories with lagrangian (A.6). These theories are distinct. For example, if both potentials are of type 1, k is not quantized. If both are of type 2, k is quantized and integral. This is the choice which appears to be most relevant to supergravity.[17] If both are of type 3, we must restrict the periods of A or of λ to be zero.

The nature of the gauge group similarly restricts the possible observables ("Wilson surfaces") and the identifications imposed between different observables by gauge symmetry. In theory 1, any function of $\int_\gamma A_i$, where γ are any closed cycles, is gauge invariant. In theory 2, the observables are

$$W(\gamma) = \exp\left[i\int_\gamma A_i\right] , \tag{A.7}$$

where γ is a cycle defining an *integral* homology class. The cycle γ generalizes the electric coupling of the gauge field, and some readers will prefer to write $\gamma = e\gamma_0$ where e is an integer and γ_0 defines a primitive integral homology class. In theory 3, there are no gauge invariant observables.

The correlation functions of these observables are easily computed. Let $\gamma_1, \ldots, \gamma_n$ be integral $(p+1)$-cycles and $\sigma_1, \ldots, \sigma_m$ be integral $(D-p-2)$-cycles. Then

$$\Big\langle W(\gamma_1)\cdots W(\gamma_n)\cdot W(\sigma_1)\cdots W(\sigma_m)\Big\rangle = \exp\left[\frac{2\pi i}{k}\sum_{r,s} L(\gamma_r,\sigma_s)\right] , \tag{A.8}$$

where $L(\gamma,\sigma)$ is the integral linking number.[18] One way to interpret this formula is that the insertion of a Wilson line along $e\gamma_0$ creates a holonomy $2\pi e/k$ for the field A_{D-p-2} around γ_0.

[17] Thus, some readers will insist that the theory (A.6) is the gauge invariant product of a dual pair of Cheeger-Simons characters. For a careful description of such products see ([41].

[18] The reader who is still awake might notice that we have neglected self-intersection terms. These require a choice of framing of the normal bundle. In the situations of interest in this paper the normal bundle will be trivial so we can take the self-intersections to be zero.

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In theory 2, the Wilson surface observables are subject to an important identification rule, namely

$$W(\gamma) \sim W(\gamma + k\gamma')\,, \tag{A.9}$$

where γ' is any cycle defining an integral homology class. This is plainly compatible with the explicit correlators (A.8) but it can also be derived by the technique of making a "singular gauge transformation," familiar from discussions of 3D Chern-Simons gauge theory, as well as from the fractional quantum Hall system. In this case, one can see that the effect of inserting a Wilson line of charge k is just performing a gauge transformation $A \to A + 2\pi d\theta$ where θ is an angle around the Wilson line to be inserted. These singular gauge transformations are allowed in the theory, they are unobservable Dirac strings.

We now describe these singular gauge transformations in the general case. Let σ be a closed integral cycle in X_D of dimension $D-p-2$. Let ζ_{p+1} be a trivialization of the poincare dual of σ on $X_D - \sigma$, i.e. a solution of $d\zeta_{p+1} = \eta(\sigma \hookrightarrow X_D)$ on $X_D - \sigma$. Thus, near X_D, ζ_{p+1} is a global angular form on the normal bundle of σ_{D-p-2}. In 3D Chern Simons theory we have $\zeta_{p+1} = d\theta$ where θ is the angle around the Dirac string. If we have a Wilson surface observable of the form $e^{ik\int_\sigma A_{D-p-2}}$ we can see from the equations of motion of (A.6) that the field configuration for A_{p+1} around it is the same as that of a singular gauge transformation of the type we have just described. Therefore the insertion of such a Wilson surface operator is unobservable, since it is equivalent to performing a singular gauge transformation, i.e. it is the same as adding an unobservable "Dirac surface."

A.3 Theory on a manifold with boundary: the singleton degrees of freedom

Let us now introduce a boundary into the theories (A.1) and (A.6).

We begin with the theory (A.1) with local degrees of freedom. Let us also begin with a "cylindrical spacetime" by which we mean a spacetime of the form $X_D = \mathbb{R}_t \times M_{D-1}$ where M_{D-1} is a manifold with boundary $\partial M_{D-1} = \Sigma_{D-2}$. One should have in mind the example of the cylinder with M_{D-1} the ball of dimension $D-1$. Let r be a normal coordinate near the boundary so that $\frac{\partial}{\partial r}$ is a unit normal vector. A system of boundary conditions such that (A.1) is a well-posed variational problem is

$$\begin{aligned} \delta A_{p+1}|_{\partial X_D} &= 0 \\ \left\{ \iota\left(\frac{\partial}{\partial r}\right)[d\lambda_p + kA_{p+1}] \right\}|_{\partial X_D} &= 0 \end{aligned} \tag{A.10}$$

(the second line is simply the normal covariant derivative). The boundary condition in the first line fixes the gauge symmetries. We can no longer make gauge transformations by Λ_p on the the boundary, and consequently, in addition to the *massive* degrees of freedom propagating in the bulk, there is a *massless* p-form field propagating along the boundary. This is closely related to the singleton degree of freedom. An analogous story holds in the dual formulation (A.5).

Let us now turn to the topological theory. In this case there are two conceptually different choices for the manifold X_D with boundary.[19]

First, if ∂X_D is compact then the path integral defines a vector in a Hilbert space. That Hilbert space is the quantization of a finite-dimensional phase space. In general the phase space is a quotient of $H^{p+1}(\partial X_D; R) \oplus H^{D-p-2}(\partial X_D; R)$. When $D = 2n+1$ and $p = n-1$ the phase space is a quotient of $H^n(\partial X_D; R)$.[20] Different vectors in the phase space are obtained by including different operators $W(\gamma)$ in the interior of X_D. Nontrivial operators on the Hilbert space are obtained from Wilson surface operators associated to cycles γ piercing the boundary ∂X_D.

Second, we can consider (A.6) on a cylindrical spacetime $X_D = \mathbb{R}_t \times M_{D-1}$. In this case the theory is holographically dual to a quantum field theory living on the boundary. These dynamical degrees of freedom on the boundary are sometimes referred to as "singleton" degrees of freedom, because of their role in AdS theories. Now, the "singular gauge transformations" act nontrivially on the Hilbert space of these theories, and should not be considered to be gauge transformations. Rather they produce modes with nontrivial fluxes for the singleton degrees of freedom. The Hilbert space includes a sum over these flux sectors. There are also new operators in the theory arising from Wilson surface operators for surfaces which pierce the boundary. These can be interpreted as insertions of charged operators in the boundary theory.

One chooses boundary conditions so that variations satisfy

$$\frac{k}{2\pi} \int_{\mathbb{R}_t \times \Sigma_{D-2}} A_{p+1} \delta A_{D-p-2} = 0\,. \tag{A.11}$$

A convenient boundary condition which satisfies (A.11) is

$$\iota\left(\frac{\partial}{\partial t}\right) A_{p+1} = \iota\left(\frac{\partial}{\partial t}\right) A_{D-p-2} = 0\,, \tag{A.12}$$

where these are the components of the gauge fields which have an overlap with the "time direction" $\mathbb{R}_t$. These boundary conditions break some of the gauge symmetry leaving unbroken the subgroup of gauge transformations which are zero on the boundary $\Sigma_{D-2} \times \mathbb{R}_t$.

[19]Cheeger-Simons characters for manifolds with boundary have been studied in [42]. In the context of theories of self-dual forms they have been studied by Witten in [43, 40, 44]. In the context of M-theory with boundary they have been studied in detail in [45].

[20]In the K-theoretic quantization of fluxes this phase space is replaced by the K-theory torus, as studied in [44, 3, 11].

In order to identify the spectrum on the boundary we follow a procedure used in [35, 46]. We note that the time components of the gauge fields in (A.6) are Lagrange multipliers, so integrating over these we impose the constraints

$$\begin{aligned} d(A_{p+1}|_{M_{D-1}}) &= 0 \\ d(A_{D-p-2}|_{M_{D-1}}) &= 0\,. \end{aligned} \tag{A.13}$$

If $H^{p+1}(M_{D-1}) = 0, H^{D-p-2}(M_{D-1}) = 0$, we can globally solve

$$A_{p+1}|_M = d^{(D-1)}\Phi_p, A_{D-p-2}|_M = d^{(D-1)}\Phi_{D-p-3}$$

and substitute back into the lagrangian.[21] The result is a total derivative, leading to a theory on the boundary of the form

$$\int_{\mathbb{R}_t\times\Sigma_{D-2}} dt \left(\frac{\partial}{\partial t}\Phi_p\right) \wedge d\Phi_{D-p-3}\,. \tag{A.14}$$

(There are several different versions of (A.14) differing by various integrations by parts.) We see that we should identify Φ_p and $d\Phi_{D-p-3}$ as conjugate variables in the sense of hamiltonian dynamics. The global nature of the gauge group is therefore the global topology of the phase space, and will affect the values that can be taken by coordinates and momenta. Canonical quantization of the theory based on the action (A.14) leads to the Hilbert space of the theory of a p form gauge field (which is the same as the theory of a $D-p-3$ form gauge field).

We can generalize the theory (A.6) by adding a surface term to the action. This term can depend on the metric on the boundary. For example, we can add to the action

$$\frac{i}{4g^2}\int_{\mathbb{R}_t\times\Sigma_{D-2}} A_{D-p-2}\wedge {}^*A_{D-p-2}\,, \tag{A.15}$$

where ${}^*A_{D-p-2}$ is the dual in the boundary. The $i=\sqrt{-1}$ in front of (A.15) shows that we view the time direction $\mathbb{R}_t$ as having lorentzian signature. Now (A.11) is replaced with

$$\int_{\mathbb{R}_t\times\Sigma_{D-2}} \left(\frac{k}{2\pi}A_{p+1} - \frac{1}{2g^2}{}^*A_{D-p-2}\right)\delta A_{D-p-2} = 0\,. \tag{A.16}$$

Free boundary conditions lead in this case to the condition

$$\frac{kg^2}{\pi}A_{p+1} = {}^*A_{D-p-2}\,. \tag{A.17}$$

[21] We are skating over several technical issues at this point. One should really introduce the entire panoply of ghosts-for-ghosts. Moreover, there are jacobian factors from the change of variables. We expect that these all cancel, but have not checked the details.

We can integrate over A_{p+1} in the bulk to find a delta functional of dA_{D-p-2} which means that A_{D-p-2} is a flat connection. Therefore, the bulk action (A.6) vanishes and the boundary action (A.15) can be written as

$$\frac{i}{4g^2}\int_{\mathbb{R}_t\times\Sigma_{D-2}} d\Phi_{D-p-3}\wedge {}^*d\Phi_{D-p-3}\,. \tag{A.18}$$

This is the standard theory of a $D-p-3$ form gauge field which can be dualized to the theory of a massless p form gauge field.

What is the difference between the theory based on (A.14) and the theory based on (A.18)? The Hilbert space of the boundary theory is that of a p form gauge field in both cases, but the hamiltonian which acts on this Hilbert space is different in the two theories. The latter depends on the details of the boundary interactions.

The case $D=4n+3$, $p=2n$ is particularly interesting. Since $p+1=D-p-2$ the two different gauge fields in (A.6) are of the same degree and we can consider two linear combinations of them A_{2n+1} and C_{2n+1} such that the action (A.6) can be written as a difference of two terms $\int A_{2n+1}dA_{2n+1}-C_{2n+1}dC_{2n+1}$. The spectrum of the theory on the boundary is that of a $2n$ form gauge field. It splits into two contributions. The selfdual part originates from A and the antiselfdual part originates from C. Therefore, we can consider the theory which is based on A only. The spectrum on the boundary of this theory is a chiral $2n$ form gauge field.

This discussion suggests a way to define an action for a chiral gauge field on a $4n+2$ dimensional space N_{4n+2}.[22] All we need to do is consider the Chern-Simons action of a $2n+1$ dimensional gauge fields in a manifold M_{4n+3} such that $\partial M_{4n+3}=N_{4n+2}$. By adding various boundary interactions we can study different hamiltonians acting on the Hilbert space of the chiral gauge field.

More explicitly, consider $N_{4n+2}=\mathbb{R}_t\times\Sigma_{4n+1}$ with lorentzian signature and study the action

$$\begin{aligned} S(A,B) &= -\frac{ik}{2\pi}\int_{M_{4n+3}} A_{2n+1}\wedge dA_{2n+1} + \\ &\quad +\frac{ik}{4\pi}\int_{N_{4n+2}} (A_{2n+1}-B_{2n+1})\wedge {}^*(A_{2n+1}-B_{2n+1}) + 2A_{2n+1}\wedge B_{2n+1} \\ &= S(A,B=0) + \frac{ik}{4\pi}\int_{N_{4n+2}} (A+{}^*A)\wedge(B-{}^*B) + B\wedge{}^*B \\ &= S(A,B=0) + \frac{ik}{4\pi}\int_{N_{4n+2}} A^{(+)}\wedge B^{(-)} + B\wedge{}^*B\,, \end{aligned} \tag{A.19}$$

where B_{2n+1} is a nondynamical $2n+1$ form background field (source) on N_{4n+2}. Its anti-selfdual part $B^{(-)}$ couples to the selfdual part of the dynamical field $A^{(+)}$.

[22] The following discussion is closely related to Witten's analysis of selfdual fields in [43, 44]. Related ideas have also been considered by S. Shatashvili.

The boundary variation including the surface term from the bulk is $[(A-B) - {}^*(A-B)]\delta A = 0$ which for free boundary conditions sets $A-B$ to a selfdual field. Performing the functional integral over A the answer is $\exp[\Gamma(B)]$ where $\Gamma(B)$ is a functional of B . Since under the gauge transformation which does not vanish on N_{4n+2} and acts also on the sources $\delta A = \delta B = d\lambda$ the action transforms as $\delta S(A,B) = \frac{ik}{2\pi}\int_{N_{4n+2}} d\lambda \wedge B$,

$$\delta\Gamma(B) = \frac{ik}{2\pi}\int_{N_{4n+2}} d\lambda \wedge B\,. \tag{A.20}$$

Therefore, Γ is a sum of a gauge invariant term and a Chern-Simons term for B on M_{4n+3}.

A.4 Examples

Let us flesh out this abstract discussion with some particular examples.

1. $D=2$ and $p=0$. We now have action $S = k\int Ad\wp$ where A is a 1-form and $\wp$ is a 0-form. Let us suppose that the gauge group is such that we identify $A \sim A+\omega$ where ω is a closed 1-form with $2\pi Z$ periods. Let us work on the strip $[-L,L]\times R_t$. If we integrate over $\wp(x,t)$ for $x \in (-L,L)$ then $A = dq$, and substituting back into the action we get the basic action of quantum mechanics

$$S = k\int dt q_1\dot{\wp}_1 - k\int dt q_2\dot{\wp}_2\,, \tag{A.21}$$

where $q_1 = q(x=L,t), \wp_1 = \wp(x=L,t), \wp_2 = \wp(x=-L,t), q_2 = q(x=-L,t)$. If we insert a "Wilson line" $\exp[ie\int A_t dt]$ then we can shift it away by a "singular gauge transformation"

$$\wp \to \wp - \frac{e}{k}\theta(x)\,. \tag{A.22}$$

Related to this, if we declare the gauge group of $\wp$ to be $\wp \sim \wp + Z$ then $\wp \to \wp + N\theta(x)$ is a gauge transformation which shifts the charge of a Wilson line by $q \to q + Nk$. Our choice of gauge group for A means $q \sim q + 2\pi Z$. So, if we also identify $\wp \sim \wp + Z$ then we are quantizing the compact phase space $S^1 \times S^1$. Then the momentum is indeed valued in Z_k. On the other hand, we could choose not to identify $\wp$. Then we are quantizing the phase space $S^1_q \times R_\wp$. Now the momenta are indeed quantized, but can take any integral value.

2. $D=3$ and $p=0$. Since $D = 4n+3$ and $p = 2n$ here we can have a single gauge field A_1. This is the famous example of 3d abelian Chern-Simons theory, which is holographically dual to the level k U(1) chiral algebra. The Wilson loop operators on the cylinder create holonomy n/k for the gauge field. The singular gauge transformations shift $n \to n+k$. These act nontrivially on the

Hilbert space associated with the boundary theory, amounting to an insertion of the extending chiral field $e^{i\sqrt{2k}\phi}$ of charge k with respect to the current. The different chiral sectors are labelled by $n \mathrm{mod} k$, and the symmetry of the fusion rules is Z_k. Nevertheless, the conformal field theory has a U(1) symmetry.

3. $D = 5, p = 1$. We have two 2-forms $B^{(1)}, B^{(2)}$. The boundary theory has a U(1) photon, $B^{(1)} = dA$, while $B^{(2)}$ gives the dual photon. The action (A.14) for this case was studied in [47].

4. $D = 7, p = 2$. This example is similar to (2) and can have a single gauge field A_4.

B. U(N) vs SU(N) in the AdS/CFT correspondence

String theory on $AdS_5 \times S^5$ is related to U(N) Yang Mills theory [48]. The U(1) part of this U(N) theory is free. In [49] and [50], it was emphasized that bulk gravity in AdS is dual to the SU(N) part of the gauge theory. It is clear from the arguments in [49, 50] that the U(1) degree of freedom cannot live in the bulk of AdS.

The theories in examples 2,3,4 in appendix A appear in the low energy description of string theory on AdS spaces. The singleton degrees of freedom play an important role in determining the precise structure of the gauge group in the holographically dual theory. As explained in appendix A, a Chern-Simons theory with a boundary leads to a degree of freedom living at the boundary. In the case of AdS_5 we have a Chern-Simons action in the bulk for the RR and NS 2-form potentials. These lead to a single U(1) gauge field on the boundary, which arises from the "gauge freedom" of the 2-form B^{NS}. (The gauge field arising from B^{RR} is the magnetic dual, see (A.17).)

Let us consider the Wilson line in representation R along a closed curve C in the boundary theory, $W_{R,C} := Tr_R Pexp \int_C \mathbf{A}$, Here $\mathbf{A}$ is the U(N) gauge field. It is an $N \times N$ antihermitian matrix-valued 1-form. Recall that an irreducible representation R of U(N) is labelled by a pair (q, λ) where q is the integral charge specifying the representation of the U(1), and λ is an irrep of SU(N) such that $q \mathrm{mod} N$ is the N-ality of λ.

In the AdS/CFT correspondence the computation of the expectation values of $W_{R,C}$ in the boundary theory is replaced by the string theory correlators of Wilson surfaces whose worldsheets Σ end on C. Such worldsheets couple to B^{NS} as $\exp[i \int_\Sigma B^{NS}]$ and therefore depend on the singleton gauge field A in the same way as a Wilson line for the U(1) subgroup of charge $q = 1$.

When C is a contractible cycle in the boundary we are computing the Wilson line associated to the creation/annihilation of a quark/antiquark pair. This was computed via AdS/CFT in [51, 52]. On the other hand, if the boundary manifold is of the form $S^1 \times M_3$ then we can instead consider a Wilson loop for $C = S^1 \times P$

where $P \in M_3$ is a point. Expectation values of

$$\mathcal{W}(P_1, \ldots, P_N) := \mathrm{Tr}_N\Big(Pexp \int_{S^1\times P_1} \mathbf{A}\Big) \cdots \mathrm{Tr}_N\Big(Pexp \int_{S^1\times P_N} \mathbf{A}\Big), \tag{B.1}$$

where P_i, $i = 1, \ldots, N$, are N points on M_3, measures the coupling to a "baryon" made out of the antisymmetric combination of N external quarks. In [27] and [50] it was argued that the existence, in the gauge theory, of a "baryon vertex," namely a gauge invariant coupling of N external quarks with nonzero expectation value, implies that the bulk physics is described by the SU(N) part of the gauge theory. We now can see that if we include the singleton mode that lives at the boundary we can also account for the U(1) degree of freedom.

At first sight the baryon vertex seems to preclude the existence of a U(1) degree of freedom living at the boundary. One argument for this is that in the path integral evaluation of (B.1) the integral over the zeromode of $\int_{S^1\times P} Tr\mathbf{A}$ would set expectation values of (B.1) and its products to zero.[23] The apparent contradiction is resolved as follows.[24] For definiteness, take $M_3 = S^3$ and fill it in with the disk D^4. The gravity dual for expectation values of (B.1) involves an insertion in the bulk of a D5-brane wrapped on $Q \times S^1 \times S^5$, where $Q \in D^4$ is a point [27]. As explained in [27] charge conservation forces us to attach N Wilson surfaces coupling to B^{NS}. However, we must also include a "Dirac string" singularity of the field B^{RR}. This "Dirac string" is actually a two dimensional surface in the five-dimensional manifold $D^4 \times S^1$, and a 7-dimensional manifold in the full spacetime, but we will refer to it as a "Dirac string." While the Dirac string for B^{RR} has no physical effect in the bulk, it *does* have a physical effect when it intersects the boundary on $S^1 \times P_0$, where P_0 is a point on S^3. Indeed, it acts as a source of charge $-N$ for the singleton mode holographically dual to B^{NS}. The reason for this is the following. The RR field around this Dirac surface is such that $\int_{S^2} B^{RR} = 2\pi$, where S^2 is the sphere linking the Dirac surface in five dimensions. As we explained above B^{NS} on the boundary is the field strength of the singleton gauge field, while B^{RR} is the field strength of the electric-magnetic dual singleton field. Then, one can either use singular gauge transformations in the BF theory $S = N \int dB^{RR} \wedge B^{NS}$, or, alternatively, invoke boundary conditions (A.17) to conclude that there is an electric field for the singleton field of strength $-N$.

The conclusion from the above reasoning is that *the holographic dual of the insertion of a wrapped $D5$ brane is the expectation value of*

$$\exp\left[-\int_{C\times P_0} \mathrm{tr}\mathbf{A}\right] \mathcal{W}(P_1, \ldots, P_N)\,. \tag{B.2}$$

[23]This issue did not arise in the computation of [51, 52], because, for a contractible C, there is no zeromode to integrate over.

[24]A second, essentially equivalent, resolution proceeds by studying the U(1) gauge invariance of an operator related to (B.1) but involving open Wilson lines.

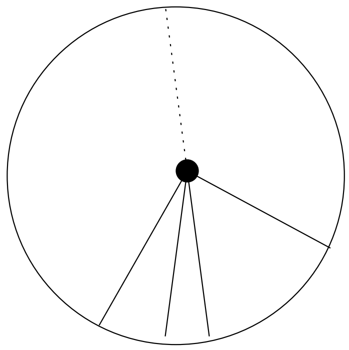

Figure 2: We consider a fixed time slice of the AdS geometry. Here we see N strings ending on a baryon vertex. A Dirac string, indicated by the doted line, emanates from the wrapped $D5$ brane. At the point where this string crosses the boundary of AdS there is an operator in the U(1) theory with charge $-N$ inserted.

That is, in terms of the gauge theory, we have N U(N) quarks inserted where the fundamental strings intersect the boundary, and we have a Wilson line which couples only to the U(1) with charge $-N$ inserted at $P_0 \in S^3$ (and winding along the S^1). Similarly, in cases where we have flux of H_{RR} on S^3 we will have Dirac surfaces intersecting S^3 at points. Again, each Dirac surface intersecting the boundary leads to the insertion of a Wilson loop for the U(1) with charge N, in the appropriate duality frame.

There is no obvious reason for (B.2) to vanish from integration over a zeromode. Moreover, the result of the computation will depend on the point P_0 where the Dirac string intersects the boundary. This dependence depends on the hamiltonian for the singleton, which in turn we can view as arising from a choice in the boundary conditions at infinity.

Even though the U(1) singleton degree of freedom has no consequence for bulk physics in AdS it is important to understand its origin in order to compare field theory answers with gravity answers. In particular, if we calculate Wilson loop expectation values in the gravity picture we need to understand, and state precisely, the boundary conditions for the B-fields on the boundary of AdS. In particular, if we do the computation by including the asymptotically flat region, then we are automatically including the U(1) degree of freedom, see for example [53, 54] for computations where it is automatically included. So when we take the decoupling limit, the finite answer that we will get will include the U(1) degrees of freedom. Similar care should be exercised when we do computations of anomalies, etc, in the bulk theory.

In the discussion of [50] a prominent role was played by

$$\alpha = \int_{D_2} B_{NS} \tag{B.3}$$

in cases where the five dimensional geometry was $D_2 \times S^3$ or $D_2 \times R^3$. It is important to understand the field theory interpretation of this variable. Consider a Wilson line $W = Tr(Pe^{\int_{S^1} \mathbf{A}})$, we define α through $W = |W|e^{i\alpha}$. When we introduce this Wilson loop operator in the path integral, via the AdS prescription, we indeed get a phase equal to (B.3). The phase $e^{i\alpha}$ depends on both the SU(N) degrees of freedom and on the U(1) degrees of freedom. If we are considering an infinite volume system, the field theory on $S^1 \times R^3$ for example, then α is fixed as a vev. In [50] the behaviour of various quantities under the SU(N) dependence of α was studied. In particular, it was argued that $\alpha \to \alpha + 2\pi/N$ should be a symmetry, but $\alpha \to \alpha + \beta$ should not be a symmetry for arbitrary β. If we perform a gauge transformation on the B field that does not vanish at infinity we see that we change $\alpha \to \alpha + \beta$, and this is equivalent to introducing a Wilson line for the singleton. In [50] these gauge transformations were not allowed so that the singleton mode was effectively frozen to zero, so that only the dependence of the physical answers on the SU(N) dependence of α were studied.

It is amusing to note that in the Klebanov-Witten [55] solution there is only one obvious set of B fields with Chern Simons terms, the same we had above, and therefore only one U(1). From the field theory point of view we could imagine starting with a U(N)$\times$U(N) theory so that we would naively expect two U(1)'s. The relative U(1) is truly decoupled in the IR by the renormalization group flow while the overall U(1) is the one we still see living at the boundary and coming from the $B-$fields in AdS. Baryons that come from D3 branes in the bulk [56] are not charged under this overall U(1). The question of whether the relative U(1) appears or not seems hard to answer.

What we have said above regarding the singleton mode in the case of $AdS_5 \times S^5$, can also be generalized to other dimensions. In the case of $AdS_7 \times S^4$ there is a three form potential C_3 in seven dimensions with a coupling $\frac{N}{4\pi}\int_{AdS_7} C_3 \times dC_3$. This C_3 leads to a self dual two form on the boundary. In the AdS_3 case, gauge fields in AdS_3, with Chern Simons couplings lead to chiral scalar fields on the boundary. In the $AdS_4 \times S^7$ case all modes living on the boundary are expected to be scalar fields and they should be modes with SO(8) quantum numbers. It would be interesting to see them.

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